 KNU/2025/BSCMTMMN401

UG 4th Semester Examination - 2025 (Under NCCF)
Award: - B.Sc
Discipline : Mathematics
Course Type : MNC-4
Course Code : BSCMTMMN401
Course Name : Abstract Algebra and Linear Algebra-II

Full Marks - 70 (Regular) Time - 3hours

1. Answer any five questions **1×5=5**

a) Define semigroup.

b) Test whether the permutation $\begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 4\,3\,1\,2\,6\,5 \end{pmatrix}$ is odd or even.

c) Give an example of a commutative subgroup of a non -commutative group.

d) Give an example of a commutative group which is not cyclic.

e) When is a matrix said to be diagonalizable?

f) Let $S = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 2\}$. Examine whether S is a subspace of the vector space $\mathbb{R}^3$.

g) Define a linear transformation.

h) State Cayley- Hamilton theorem.

2. Answer any ten questions **2×10= 20**

a) If in a group $(G,*)$, $a*c = b*c$ holds for all $a,b,c,\in G$, then prove that $a = b$.

b) Show that a group G is abelian if $(ab)^2 = a^2 b^2$, $\forall a, b \in$G.

c) Give an example of an infinite group of which every element is of finite order.

d) Show that the set E of even integers is a subgroup of the additive group (Z,+) of integers.

e) If G is a group and H is a subgroup of index 2, prove that H is a normal subgroup of G.

f) Prove that any field is an integral domain.

g) Show that the ring of integers (Z, +,.) is not a field.

h) Show that the vectors {(2,–3,1), (3–1, 5), (1,–4,3)} are linearly independent.

i) Show that the set S{(1,0,0), (1,1,0) , (1,1,1) (0,1,0)} spans the vector space $\mathbb{R}^3$, but is not a basis.

j) Determine the subspace of $\mathbb{R}^3$ spanned by the vectors (1,2,3), (3,1,0).

k) Let $V = P_2(IR)$, the vector space of all real polynomials of degree at most 2. Prove that the mapping D: $V \rightarrow V$ defined by $Df(x) = \frac{d}{dx} f(x), f(x) \in V$ is a linear mapping.

l) Let V and W be vector spaces over a filed F and $T : V \rightarrow W$ be a linear mapping. Prove that $T(\theta) = \theta'$, where θ and θ' are null elements of V and W respectively.

m) Define Kernel of a linear transformation.

n) Prove that the linear mapping $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $T(x, y, z) = (x + y, y + z, z + x), (x, y, z) \in \mathbb{R}^3$ is one -to -one and onto.

3. **Answer any five questions.** **5×5=25**

a) (i) Define Dihedral group with an example.

(ii) Give an example of an abelian group which is not cyclic 3+2

b) Let G be a group. Prove that (i) $(a^{-1})^{-1} = a, \forall a \in G$ (ii) $(ab)^{-1} = b^{-1}a^{-1}, \forall a, b \in G$. 2+3

c) Prove that the roots of the equation $x^6 - 1 = 0$ form a subgroup of the multiplicative group of non-zero complex numbers. Is the subgroup cyclic? Justify your answer.

d) Show that (Z,+,.) is a ring with respect usual addition and scalar multiplication.

e) Diagonalize the matrix : $A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$.

f) Find a basis and dimension of the subspace W of $\mathbb{R}^3$, where

$W = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$.

g) Find eigenvalues and eigenvectors of the following matrix : $\begin{pmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

h) A mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x,y,z) = (x+y+z, 2x+y+2z, x+2y+z)$, $(x,y,z) \in \mathbb{R}^3$. Prove that T is a linear mapping. Find Ker T and the dimension of Ker T.

4. **Answer any two questions :** **10×2= 20**

a) (i) Show that the set $S = \{a + b\omega : a, b \in \mathbb{R}\}$ forms a field with respect to usual addition and multiplication of complex numbers, where ω is the cube root of unity and $\mathbb{R}$ is the set of all real numbers.

(ii) If (G, o) be a finite group of even order. Prove that G contains an odd number of elements of order 2.

5+5

b) i) Prove that every subgroup of a commutative group G is a normal subgroup of G.

ii) Prove that the intersection of two normal subgroups of a group G is normal in G.

iii) Show that the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ is not diagonalisable. 3+3+4

c) i) Determine the linear mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$ which maps the basis vectors

(0,1,1), (1,0,1), (1,1,0) of $\mathbb{R}^3$ to the vectors (2,0,0), (0, 2, 0), (0, 0, 2) respectively. Find Ker T and Im T. Verify that *dim Ker T* + *dim Im T* =3 .

ii) Determine whether the vectors { (1,1,2), (3,4,7), (5,3,1)} form a basis of $\mathbb{R}^3$ or not.

iii) Use Cayley-Hamilton theorem to find A^{-1}, where $A = \begin{pmatrix} 2 & -7 \\ -1 & 3 \end{pmatrix}$. 5+3+2

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